DIFFERENTIAL OPERATORS ON AZUMAYA ALGEBRA AND HEISENBERG ALGEBRA

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1. Introduction

Let $\mathbb{k}$ be a field and A an associative $\mathbb{k}$ -algebra. Let $\bigotimes$ mean $\bigotimes_{\mathbb{k}}$ and $A^e := A \bigotimes A^o$, where A^o denotes the opposite algebra of A. Any left A^e-module M is the same as an A-bimodule given by $a{\cdot}m{\cdot}b = (a \otimes b^o){\cdot}m$ for $m \in M$ and $a, b \in A$. In [7], V. Lunts and A. Rosenberg give a definition for the ring of differential operators on a left A-module L (denoted by $D_{\mathbb{k}}({}_AL)$) and the A-bimodule of differential operators on L of order less than or equal to m denoted by $D^m_{\mathbb{k}}({}_AL)$. When $L = A$, we denote the ring and module respectively by $D_{\mathbb{k}}(A)$ and $D^m_{\mathbb{k}}(A)$. These definitions are recalled in the section of preliminaries in this paper.

In this paper we compute the ring of differential operators for some noncommutative rings, namely the Axumaya algebras and the Heisenberg algebras. The initial interest was to compute these rings for matrix algebras and the Weyl algebras, which were easily generalized.

We consider Azumaya algebras over a noetherian ring. We conclude that the ring of differential operators are generated as modules by the ring of differential operators on their centre and homomorphisms given by multiplication by elements of the bigger ring (called inner homomorphisms). That is, if R is the centre of A (where A is an Azumaya algebra over R), we show that $D_{\mathbb{k}}(R)$ can be embedded into $D_{\mathbb{k}}(A)$ and that $D_{\mathbb{k}}(A) = (A \bigotimes_R A^o).D_{\mathbb{k}}(R).(A \bigotimes_R A^o)$, that is, $D_{\mathbb{k}}(A)$ is generated as an A-bimodule by $D_{\mathbb{k}}(R)$ (Theorem 3.2.5).

In the case of Heisenberg algebras of zero characteristic, we need two copies of differential operators on the centre to generate all the differential operators (Theorem 4.1.9). The non zero characteristic follows from the study on Azumaya algebras, because in this case the Heisenberg algebra is Azumaya over its centre (Theorem 2 [8]).

In particular, our work on these general rings show that

1. If R is a $\mathbb{k}$ -algebra, we show that $D_{\mathbb{k}}(M_n(R)) = M_{n^2}(D_{\mathbb{k}}(R))$, where $M_n(R)$ denotes the algebra of $n \times n$ matrices over R (Corollary 3.1.3).

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2. If A_n denotes the n-th Weyl algebra over a field of characteristic 0, then $D_{\mathbb{k}}(A_n) = A_{2n}$ (Corollary 4.1.8).

In the case of Azumaya algebras, we show that there is a one-to-one correspondence between ideals of $D_{\mathbb{k}}(A)$ and $D_{\mathbb{k}}(R)$ (section 3.3). If H_n denotes the nth- Heisenberg algebra, we show that $D_{\mathbb{k}}(H_n)$ is simple (Theorem 4.1.10 and corollary 4.2.2).

We give some definitions and prove some elementary results in the section of preliminaries. These results will be used later, and are interesting in their own right. This will be followed by a section on the differential operators on the Azumaya algebras. Here, we first show that if A is an Azumaya algebra over R, then $D^m_{\mathbb{k}}(A) = D^m_{\mathbb{k}}({}_RA)$ (Theorem 3.1.1) for each $m \geq 0$. Then we show that $D_{\mathbb{k}}(R)$ embeds as an R-bimodule in $D_{\mathbb{k}}({}_RA)$ (respecting the filtration given by the order of differential operators), and along with the inner differential operators, generate the entire ring $D_{\mathbb{k}}({}_RA)$ (Theorem 3.2.5).

The last section covers the Heisenberg algebras. We consider the two cases of zero characteristic and non zero characteristic separately.

Notations :

1. For any $\mathbb{k}$-algebra S, and $s \in S$, denote by $\lambda_s \in Hom_{\mathbb{k}}(S, S)$ (respectively ρ_s) to be the homomorphism given by left-multiplication (respectively, right-multiplication) by s.
2. For any $\varphi \in Hom_{\mathbb{k}}(S, S)$, let $[\varphi, s] := \varphi \cdot s - s \cdot \varphi$.
3. For $r, s \in S$, let $[r, s] := rs - sr$.

2. Preliminaries

We recall from [7] some definitions.

Definition 2.0.1.

1. *For an A^e module M, its centre is the $\mathbb{k}$ -submodule*
$$\mathcal{Z}(M) := \{z \in M | a \cdot z = z \cdot a \text{ for } a \in A\}.$$
2. *Define the i-th differential part of M, $\mathcal{Z}_iM$ by induction as follows:*
$$\mathcal{Z}_0M := A^e\mathcal{Z}(M), \text{ and}$$
$$\mathcal{Z}_iM / \mathcal{Z}_{i-1}M := A^e\mathcal{Z}\left(M / \mathcal{Z}_{i-1}M\right) \text{ for } i \geq 1.$$
3. *The differential part of an A^e module M is $M_{diff} := \cup_{i\geq 0}\mathcal{Z}_iM$.*

For L a left A-module, the $\mathbb{k}$ -vector space $Hom_{\mathbb{k}}(L, L)$ has an A^e-module structure given by $c \cdot \varphi \cdot a(b) = c\varphi(ab)$ for $a, b, c \in A$ and $\varphi \in Hom_{\mathbb{k}}(L, L)$. The *differential part* of $Hom_{\mathbb{k}}(L, L)$ is the *algebra of $\mathbb{k}$ -linear differential operators on* L, and denoted by $D_{\mathbb{k}}({}_AL)$. The A^e module of differential operators of order $\leq m$ on A is $\mathcal{Z}_mHom_{\mathbb{k}}(L, L)$

and is denoted by $D^m_{\Bbbk}({}_AL)$. This definition generalizes the definition of differential operators on a commutative ring as given by Grothendieck ([5]). We denote $D_{\Bbbk}({}_AA)$ simply by $D_{\Bbbk}(A)$.

We state and prove some preliminary results.

Proposition 2.0.2. *For any ring A, the ring $D^0_{\Bbbk}(A)$ consists of left and right multiplications by elements of A.*

Proof. The central elements of the A^e-module $Hom_{\Bbbk}(A, A)$ are homomorphisms given by right multiplication by elements of A. Hence the result. □

Corollary 2.0.3. *There is a surjection*

$$A \bigotimes_{\mathcal{Z}(A)} A^o \to D^0_{\Bbbk}(A) \text{ given by}$$
$$a \otimes b^o \mapsto [c \mapsto acb],$$

where $\mathcal{Z}(A)$ is the centre of the ring A.

Proposition 2.0.4. *Let $R \subset S$ be two $\Bbbk$ -algebras, and M be an S^e-module (hence an R^e-module). If $R \subset \mathcal{Z}(S)$, the centre of S, then the i-th differential part of M considered as an S-bimodule is contained in the i-th differential part of M considered as an R-bimodule.*

Proof. For any S-bimodule N we have,

$$S \cdot \mathcal{Z}_S(N) \subset \mathcal{Z}_R(N),$$

where $\mathcal{Z}_S(N)$ denotes the S-centre of N (analogously defined for R). Hence the proposition. □

Corollary 2.0.5. *Let $R \subset S$ be two $\Bbbk$ -algebras. If $R \subset \mathcal{Z}(S)$, then $D^m_{\Bbbk}(S) \subset D^m_{\Bbbk}({}_RS)$ for $m \geq 0$.*

Remark 2.0.6. *The corollary above is not true if $R \not\subseteq \mathcal{Z}(S)$. Consider for example,*

$$R = \Bbbk[x] \subset S = \Bbbk < x, y > \diagup < [y, x] = y > .$$

Note that R is commutative. Hence, $\varphi \in D_{\Bbbk}({}_RS)$ satisfies

$$[[\cdots[\varphi, r_1], r_s], \cdots r_n] = 0,$$

for some $n \geq 0$. We know that λ_y, the homomorphism given by left multiplication by y is in $D_{\Bbbk}(S)$. But $[\lambda_y, x] = \lambda_y$. Hence $\lambda_y \notin D_{\Bbbk}({}_RS)$.

Let L be a free, left-R-module, where R is a commutative $\Bbbk$ -algebra. Fix a basis $\{l_1, l_2, \cdots, l_n\}$ of L over R. Any $\Phi \in Hom_{\Bbbk}(A, A)$ can be written as

$$\Phi = \begin{array}{c c} & \begin{array}{cccc} R.l_1 & R.l_2 & \cdots & R.l_n \end{array} \\ \begin{array}{c} R.l_1 \\ R.l_2 \\ \vdots \\ R.l_n \end{array} & \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots & \varphi_{1,n} \\ \varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n,1} & \varphi_{n,2} & \cdots & \varphi_{n,n} \end{pmatrix} \end{array} \tag{2.0.1}$$

where $\varphi_{i,j} \in Hom_{\Bbbk}(R, R)$.

Proposition 2.0.7. *Referring to the equation 2.0.1, $\Phi \in D^m_{\Bbbk}({}_RL)$ if and only if $\varphi_{i,j} \in D^m_{\Bbbk}(R)$.*

Proof. Since R is commutative,
$\Phi \in D^m_{\Bbbk}({}_RA)$ (respectively, $\varphi \in D^m_{\Bbbk}(R)$), if and only if $[\cdots[[\Phi, r_0], r_1], \cdots, r_m] = 0$ (respectively, $[\cdots[[\varphi, r_0], r_1], \cdots, r_m] = 0$), for $r_0, r_1, \cdots, r_m \in R$. The proposition follows immediately once we notice that if Φ is given by a matrix $(\varphi_{i,j})$, then $[\Phi, r]$ is given by the matrix $([\varphi_{i,j}, r])$ for $r \in R$. $\square$

3. Differential operators on Azumaya algebras

Let R be a commutative, Noetherian $\Bbbk$ -algebra. Let A be an Azumaya algebra over R (see [3] for a complete study); i.e., A is an R-algebra which is finitely generated, projective, and faithful as an R-module, such that $R \cdot 1 = \mathcal{Z}(A)$ and the map

$$A \bigotimes_R A^o \to Hom_R(A, A),$$
$$a \otimes b^o \mapsto [c \mapsto acb]$$

is an isomorphism. Examples are matrix algebras over R. Some immediate remarks follow:

Remark 3.0.8. $D^0_{\Bbbk}(A) = Hom_R(A, A) \cong A \bigotimes_R A^o$*, and hence* $D^0_{\Bbbk}(A) = D^0_{\Bbbk}({}_RA)$*. Indeed, referring to the corollary 2.0.3, there is a surjection*

$$A \bigotimes_R A^o \to D^0_{\Bbbk}(A) \tag{3.0.2}$$

On the other hand, since A is an Azumaya algebra, $A \bigotimes_R A^o \cong Hom_R(A, A)$ *given by the map* $a \otimes b^o(c) = acb$*. Thus,* $A \bigotimes_R A^o$ *injects into* $Hom_{\Bbbk}(A, A)$*. Hence, the surjection 3.0.2 onto* $D^0_{\Bbbk}(A)$ *is an isomorphism. By definition,* $D^0_{\Bbbk}({}_RA) = Hom_R(A, A)$*.*

Remark 3.0.9. *By corollary 2.0.5, for each $m \geq 0$, we have a map of R-bimodules, namely*

$$\iota_A : D^m_{\Bbbk}(A) \hookrightarrow D^m_{\Bbbk}({}_R A). \tag{3.0.3}$$

3.1. Proof of $D_{\Bbbk}({}_R A) = D_{\Bbbk}(A)$.

Theorem 3.1.1. *The inclusion of 3.0.3 is an isomorphism. That is, for each $m \geq 0$, we have $D^m_{\Bbbk}(A) = D^m_{\Bbbk}({}_R A)$ as R-bimodules.*

Proof. We first prove the theorem in the case when A is free over R with basis $\{a_1, a_2, \cdots, a_n\}$. By proposition 2.0.7, any $\Phi \in D^m_{\Bbbk}({}_R A)$ if and only if all the $\varphi_{i,j} \in D^m_{\Bbbk}(R)$. It remains to show that if all the $\varphi_{i,j} \in D^m_{\Bbbk}(R)$, then $\Phi \in D^m_{\Bbbk}(A)$. Let $a_i \cdot a_j = \sum_k r^k_{i,j} a_k$. For any $\varphi \in Hom_{\Bbbk}(R, R)$, define $\tilde{\varphi} \in Hom_{\Bbbk}(A, A)$ as $\tilde{\varphi}(r a_i) = \varphi(r) a_i$. For each $1 \leq l, k \leq n$, define $\wp^{l,k} \in Hom_R(A, A)$ (and hence in $D^0_{\Bbbk}(A)$) given by $\wp^{l,k}(a_i) = \delta_{i,k} a_l$. Then, we have $\Phi = \sum_{i,j} \wp^{i,1} \widetilde{\varphi_{i,j}} \wp^{1,j}$. Thus, it remains to show that if $\varphi \in D^m_{\Bbbk}(R)$, then $\tilde{\varphi} \in D^m_{\Bbbk}(A)$. Using induction on m and the following identity,

$$[\tilde{\varphi}, a_j] = \sum_{i,k} \widetilde{[\varphi, r^k_{j,i}]} \wp^{k,i}$$

we conclude the theorem in the case when A is free as an R-module.

In the case when A is not free as an R-module, we consider the localization of A with respect to a prime ideal P of R. By Lemma 5.1, pg61 of [3], $A_P := R_P \bigotimes_R A$ is an Azumaya R_P-algebra. Consider the injective (by flatness of R_P as an R-module) map

$$id \otimes \iota_A : R_P \bigotimes_R D^m_{\Bbbk}(A) \to R_P \bigotimes_R D^m_{\Bbbk}({}_R A). \tag{3.1.1}$$

By Proposition 16.8.6 of [5], $R_P \bigotimes_R D^m_{\Bbbk}({}_R A) \cong D^m_{\Bbbk}({}_{R_P} A_P)$ which by our discussion on the free Azumaya case is isomorphic to $D^m_{\Bbbk}(A_P)$. Thus, it is sufficient to show that the inclusion of equation 3.1.1 $id \otimes \iota_A : R_P \bigotimes_R D^m_{\Bbbk}(A) \to D^m_{\Bbbk}(A_P)$ is surjective. The following lemma proves this which completes the theorem. □

Lemma 3.1.2. *For $m \geqslant 0$, the map*

$$id \otimes \imath_A : R_P \bigotimes_R D^m_{\Bbbk}(A) \longrightarrow D^m_{\Bbbk}(A_P)$$

is surjective.

Proof. We prove both the statement by induction on m. Let

$$x = \sum_i (a_i/s_i) \otimes (b_i/t_i)^o \in (A_P \bigotimes_{R_P} A^o_P) \cong D^0_{\Bbbk}(A_P)$$

be given. There is an s in $R \setminus P$, such that $sx = \sum_i a_i' \otimes (b_i')^o \in D^0_{\Bbbk}(A)$. Thus, $(1/s) \otimes sx \in R_P \bigotimes_R D^0_{\Bbbk}(A)$ is mapped to x under $id \otimes \imath_A$. So, the result is proved for $m = 0$.

Assuming that the proposition is proved for m (which implies that $D^m_{\Bbbk}(A) = D^m_{\Bbbk}({}_R A)$), let

$$d \in R_P \bigotimes_R D^{m+1}_{\Bbbk}({}_R A) = D^{m+1}_{\Bbbk}({}_{R_P} A_P) = D^{m+1}_{\Bbbk}(A_P),$$

be such that $(a/s) \cdot d - d \cdot (a/s) \in D^m_{\Bbbk}(A_P)$ for every $(a/s) \in A_P$. It is enough to show that d is in the image of $(id \otimes \imath_A)$. Note, $sd \in D^{m+1}_{\Bbbk}({}_R A)$ for some $s \in R \setminus P$. Let $\{a_1, a_2, \cdots, a_n\}$ be a finite set of generators of A as an R-module. $(a_i/1) \cdot (sd) - (sd) \cdot (a_i/1) \in D^m_{\Bbbk}({}_{R_P} A_P)$. By induction hypothesis, for each i, $1 \leqslant i \leqslant n$, there exists a t_i in $R \setminus P$, such that $t_i \cdot [a_i \cdot (sd) - (sd) \cdot a_i] \in D^m_{\Bbbk}(A)$. Let, $t = t_1 t_2 \cdots t_n$. Then, $[a_i \cdot (tsd) - (tsd) \cdot a_i] \in D^m_{\Bbbk}(A)$, for all i, $1 \leqslant i \leqslant n$. Let $ts = T \in R$. For, $r \in R$ and a_i a generator, consider $(ra_i) \cdot (Td) - (Td) \cdot (ra_i) = r[a_i \cdot (Td) - (Td) \cdot a_i] + [r \cdot (Td) - (Td) \cdot r] \cdot a_i$. Now, $r[a_i \cdot (Td) - (Td) \cdot a_i] \in D^m_{\Bbbk}(A)$. Since $sd \in D^{m+1}_{\Bbbk}({}_R A)$, $Td \in D^{m+1}_{\Bbbk}({}_R A)$, which implies $[r \cdot (Td) - (Td) \cdot r] \in D^m_{\Bbbk}({}_R A)$. But, by induction hypothesis, $D^m_{\Bbbk}({}_R A) = D^m_{\Bbbk}(A)$. Hence, $[r \cdot (Td) - (Td) \cdot r] \cdot a_i \in D^m_{\Bbbk}(A)$. Hence, for any $a \in A$, $a \cdot (Td) - (Td) \cdot a \in D^m_{\Bbbk}(A)$. Thus, $(Td) \in D^{m+1}_{\Bbbk}(A)$. Hence, $d \in R_P \bigotimes_R D^{m+1}_{\Bbbk}(A)$. This proves the lemma. □

Corollary 3.1.3. *Let $M_n(R)$ denote the ring of matrices over R where R is a commutative $\Bbbk$ -algebra. Then,*

$$D^m_{\Bbbk}(M_n(R)) = M_{n^2}(D^m_{\Bbbk}(R)), \text{ and hence}$$
$$D_{\Bbbk}(M_n(R)) = M_{n^2}(D_{\Bbbk}(R)).$$

Proof. The theorem above shows that $D^m_{\Bbbk}(M_n(R)) = D^m_{\Bbbk}({}_R M_n(R))$. The ring $M_n(R)$ is free as a left R-module. By Proposition 2.0.7 the corollary is proved. □

Remark 3.1.4. *In* [6] *we have proved a more general statement. If R and S are two $\Bbbk$ -algebras such that S is finite dimensional as a $\Bbbk$ -vector space, then $D_{\Bbbk}(R \bigotimes S) = D_{\Bbbk}(R) \bigotimes D_{\Bbbk}(S)$.*

3.2. $D_{\Bbbk}(A)$ **is generated by** $D_{\Bbbk}(R)$ **and inner homomorphisms.** Here we embed $D^m_{\Bbbk}(R)$ (as R-bimodules) into $D^m_{\Bbbk}({}_R A)$ for each $m \geq 0$.

By Lemma 3.1 of [3], R is an R-direct summand of A; that is, $A \cong R \oplus B$ as left R-modules. Fix one such decomposition. Since A is projective as a left R-module, B is also a projective as a left R-module. By assumption, R is a Noetherian ring. Hence B is a finitely generated

R-module. By the Dual Basis Lemma (lemma 1.3 of [3]), we choose a a collection $\{b_i, f_i\}_{1\leq i\leq n}$, where $b_i \in B$ and $f_i \in Hom_R(A, R)$ (we can consider f_i to be elements of $Hom_R(A, A)$ by the natural inclusion of R into A) such that $b = \sum_i f_i(b)b_i$ for $b \in B$. Let $f_0 \in Hom_R(A, A)$ be the projection of A onto R, and $b_0 = 1$. Extend f_i for $i \geq 1$ to A by defining $f_i(r) = 0$ (we denote the extension also by f_i). Then the collection $\{b_i, f_i\}_{0\leq i\leq n}$ is a dual basis of A.

Remark 3.2.1. *By definition, for $0 \leq i \leq n$, the homomorphisms f_i are differential operators of order 0. That is, they are inner homomorphisms.*

We describe a way to extend elements of $Hom_{\Bbbk}(R, R)$ to that of $Hom_{\Bbbk}(A, A)$.

Definition 3.2.2. *For $\varphi \in Hom_{\Bbbk}(R, R)$, define $\bar{\varphi} \in Hom_{\Bbbk}(A, A)$ as*

$$\overline{\varphi} = \sum_{i=0}^{n} \rho_{b_i} \varphi f_i,$$

where ρ_{b_i} is the homomorphism given by right multiplication by b_i.

Since $a = \sum_{i\geq 0} f_i(a)b_i$, we have, $\overline{id} = id$ where id denotes the identity homomorphism in the respective rings. An immediate consequence is the following lemma.

Lemma 3.2.3. *If $\varphi \in D^m_{\Bbbk}(R)$, then $\overline{\varphi} \in D^m_{\Bbbk}(A)$.*

Proof. It is clear to see that $\overline{s \cdot \varphi \cdot r} = s \cdot \overline{\varphi} \cdot r$ for $r, s \in R$ and $\varphi \in Hom_{\Bbbk}(R, R)$. Thus, $\varphi \in D^m_{\Bbbk}(R)$ implies that $\overline{\varphi} \in D^m_{\Bbbk}({}_RA)$. Now use theorem 3.1.1 to complete the lemma. $\square$

Remark 3.2.4. *By choice of f_i, we have $\overline{\varphi}(r) = \varphi(r)$ for $r \in R$. Hence the association $\varphi \mapsto \overline{\varphi}$ is an injective map of R-bimodules.*

Theorem 3.2.5. *$D^m_{\Bbbk}(A)$ is generated as an A-bimodule by $\{\overline{\varphi} | \varphi \in D^m_{\Bbbk}(R)\}$; that is,*

$$D^m_{\Bbbk}(A) = (A \bigotimes_R A^o) \cdot D^m_{\Bbbk}(R) \cdot (A \bigotimes_R A^o).$$

Proof. Let $\Phi \in D^m_{\Bbbk}(A)$. For each $i, j \in \{0, 1, 2, \cdots, n\}$ let

$$(\Phi)_{i,j} = f_i \Phi \rho_{b_j}. \tag{3.2.1}$$

Note that $(\Phi)_{i,j}(R) \subset R$. For $r, s \in R$, we see that

$$r \cdot (\Phi)_{i,j} \cdot s = (r \cdot \Phi \cdot s)_{i,j}.$$

Hence, $(\Phi)_{i,j} \in D^m_{\Bbbk}(R)$. Now, by the dual basis lemma,

$$\begin{aligned}\Phi(a) &= \sum_{j\geq 0} \Phi(f_j(a)b_j) \\ &= \sum_{i,j\geq 0} f_i(\Phi(f_j(a)b_j))b_i.\end{aligned}$$

Hence,

$$\begin{aligned}\Phi &= \sum_{i,j\geq 0} \rho_{b_i} f_i \Phi \rho_{b_j} f_j \\ &= \sum_{i,j\geq 0} \rho_{b_i} (\Phi)_{i,j} f_j \end{aligned} \tag{3.2.2}$$

Since $f_j(A) \subset R$, we have $(\Phi)_{i,j} f_j = \overline{(\Phi)_{i,j}} f_j$. Hence, equation 3.2.2 gives

$$\Phi = \sum_{i,j\geq 0} \rho_{b_i} \overline{(\Phi)_{i,j}} f_j \in (A \bigotimes_R A^o) \cdot \overline{D^m_{\Bbbk}(R)} \cdot (A \bigotimes_R A^o).$$

Hence the theorem. $\square$

3.3. **Ideals of $D_{\Bbbk}(A)$.** In this section, we show a one to one correspondence between ideals of $D_{\Bbbk}(A)$ and of $D_{\Bbbk}(R)$. Let $\mathcal{I}_A$ and $\mathcal{I}_R$ denote the collection of ideals in A and R respectively.

Lemma 3.3.1. *For $I \in \mathcal{I}_A$, the set $f_0 I f_0$ is an ideal in $D_{\Bbbk}(R)$.*

Proof. The lemma follows from the fact that, for $\varphi_1, \varphi_2 \in D_{\Bbbk}(R)$, and $\Phi \in D_{\Bbbk}(A)$, we have $\varphi_1 f_0 \Phi f_0 \varphi_2 = f_0 \overline{\varphi_1} \Phi \overline{\varphi_2} f_0$. $\square$

Define functions ζ and η as follows:

$$\begin{aligned}\zeta : \mathcal{I}_A &\to \mathcal{I}_R \\ I &\mapsto f_0 I f_0 \end{aligned} \tag{3.3.1}$$

$$\begin{aligned}\eta : \mathcal{I}_R &\to \mathcal{I}_A \\ J &\mapsto D_{\Bbbk}(A) \overline{J} D_{\Bbbk}(A) \end{aligned} \tag{3.3.2}$$

Theorem 3.3.2. *The correspondence ζ is a bijective function from $\mathcal{I}_A$ to $\mathcal{I}_R$ with η being its inverse function.*

Proof. We show that $\eta\zeta$ is the identity on $\mathcal{I}_A$. Clearly, $\eta(\zeta(I)) \subset I$. Let any $\Phi \in I$. The $(\Phi)_{i,j}$ as defined in 3.2.1 are in $I \cap f_0 I f_0$. Referring to equation 3.2.2, the claim is proved.

Now we show that $\zeta\eta$ is the identity on $\mathcal{I}_R$. Again, it is obvious that $J \subset \zeta\eta(J)$. For the reverse inclusion, use the fact that

$$D_{\Bbbk}(A) = (A \bigotimes_R A^o)\overline{D_{\Bbbk}(R)}(A \bigotimes_R A^o).$$

For $a, b, c, d, p, q, m, n \in A$, $\psi_1, \psi_2 \in D_{\Bbbk}(R)$ and $\varphi \in J$, we see that

$$\begin{aligned} & f_0 \left((p \otimes q^o)\overline{\psi_2}(c \otimes d^o)\overline{\varphi}(a \otimes b^o)\overline{\psi_1}(m \otimes n^o)\right) f_0 \\ & \quad = \sum_{i,j,k\geq 0} (f_0(pb_kq) \cdot \psi_2 \cdot f_k(cb_id)) \circ (\varphi \cdot f_i(ab_jb)) \circ (\psi_1 \cdot f_j(mn)) \\ & \quad \in J. \end{aligned}$$

Hence the theorem. □

Corollary 3.3.3. *The ring $D_{\Bbbk}(A)$ is noetherian if and only if the ring $D_{\Bbbk}(R)$ is.*

Corollary 3.3.4. *A ring S is called Prime if for any ideals P, Q of S, if $PQ = 0$ and $P \neq 0$, then $Q = 0$. The ring $D_{\Bbbk}(A)$ is prime if and only if $D_{\Bbbk}(R)$ is.*

4. Differential operators on the Heisenberg algebras

Let $\Bbbk$ be a field and n a positive integer. Let H_n denote the nth-Heisenberg algebra over $\Bbbk$. That is, H_n is a $\Bbbk$-algebra with generators $h, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ such that $[x_i, y_j] = \delta_{i,j}h$ and all the other commutators between the generators equal 0.

In this section, we show that the ring of differential operators on H_n is generated by two copies of $D_{\Bbbk}(R)$ in the case of zero characteristic and one copy in the non zero characteristic, where R denotes the centre of H_n. Note that in the case of non zero characteristic, the centre is very large (that is, H_n is free of finite rank as a module over its centre).

4.1. **Characteristic of $\Bbbk$ is 0.** In this case, the centre of H_n is $\Bbbk[h]$, the polynomial ring in one variable. Here, we have two different inclusions of $D_{\Bbbk}(\Bbbk[h])$ into $D_{\Bbbk}(H_n)$.

Let $I = (i_1, i_2, \cdots, i_n) \in (\mathbb{Z}_+)^n$ be a multi index. Denote by $\mathbf{x}^I$ the element $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$. Note that every element $a \in H_n$ can be written uniquely as $a = \sum_{I,J} p_{I,J}(h)\mathbf{x}^I\mathbf{y}^J$ where I, J are multi indices in $(\mathbb{Z}_+)^n$ and $p_{I,J}(h)$ is a polynomial in h with coefficients in $\Bbbk$.

For a multiindex $I = (i_1, i_2, \cdots, i_n)$, let $|I|$ denote the sum $(i_1 + i_2 + \cdots + i_n)$. We define two kinds of degree on H_n.

1. For $a \in H_n$ such that $a = \sum p_{I,J}\mathbf{x}^I\mathbf{y}^J$ define deg_1 of a as
$$deg_1(a) = max\{|I| + |J| \mid p_{I,J} \neq 0\}.$$
2. Define $deg_2(x_i) = deg_2(y_i) = 1$ and $deg_2(h) = 2$ and extend this degree to the entire ring.

We see that H_n is filtered as $H_n = \cup_{k\geq 0} H_n^k$ where
$$H_n^k = \{a | deg_1(a) \leq k\}.$$

Note 4.1.1. *For $a \in H_n^k$, and $r \in \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m\}$, we have $[r, a] \in H_n^{k-1}$.*

Lemma 4.1.2. *For any H_n-bimodule M, let $m \in M_{diff}$ (as defined in definition 2.0.1). Then there exists a $k \geq 0$ such that $[\cdots[[m, r_1], r_2], \cdots, r_k] = 0$ for $r_i \in \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\}$.*

Proof. Let $m \in \mathcal{Z}_t M$ (definition 2.0.1) for some $t \geq 0$ such that $m = a.n$ for some $a \in H_n$ and $n \in \mathcal{Z}(M/\mathcal{Z}_{t-1}M)$. It is enough to show that there exists an $l \geq 0$ such that $[\cdots[[m, r_0], r_1], \cdots, r_l] \in \mathcal{Z}_{t-1}M$ for $r_i \in \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\}$. If $a \in \Bbbk[h]$ then $l = 1$. Else, $a \in H_n^l$ for some $l \geq 0$. By referring to the note 4.1.1, we have the lemma. □

Corollary 4.1.3. *Let $\varphi \in D_{\Bbbk}(H_n)$. Then there exists a $k \geq 0$ such that $[\cdots[[\varphi, r_0], r_1], \cdots, r_k] = 0$ for $r_i \in \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\}$. Note that a $\varphi \in Hom_{\Bbbk}(H_n, H_n)$ satisfying this condition is in $D_{\Bbbk}(H_n)$.*

The corollary above provides another filtration of $D_{\Bbbk}(H_n)$ given by $D_{\Bbbk}(H_n) = \cup_{l\geq 0} M_l$ where
$$M_l = \{\varphi \in D_{\Bbbk}(H_n) | [\cdots[[\varphi, r_0], r_1], \cdots, r_l] = 0\}$$
for $r_i \in \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\}$.

Note 4.1.4. *M_l is closed under $+$ and $[M_l, h] \subset M_{l-2}$.*

Lemma 4.1.5. $M_l \cdot M_s \subsetneqq M_{l+s}$

Proof. Immediate once we see that $[\varphi_1\varphi_2, r] = \varphi_1[\varphi_2, r] + [\varphi_1, r]\varphi_2$. □

Definition 4.1.6. *Let $\partial_{x_l}, \partial_{y_l}, \partial_h, \overline{\partial_h} \in Hom_{\Bbbk}(H_n, H_n)$ be defined as*

$$\partial_{x_l}(px_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y_1^{j_1}y_2^{j_2}\cdots y_n^{j_n}) = i_l p x_1^{i_1}x_2^{i_2}\cdots x_l^{i_l-1}\cdots x_n^{i_n}y_1^{j_1}y_2^{j_2}\cdots y_n^{j_n},$$
$$\partial_{y_l}(px_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y_1^{j_1}y_2^{j_2}\cdots y_n^{j_n}) = j_l p x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y_1^{j_1}y_2^{j_2}\cdots y_l^{j_l-1}\cdots y_n^{j_n},$$
$$\partial_h(px_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y_1^{j_1}y_2^{j_2}\cdots y_n^{j_n}) = p' x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}y_1^{j_1}y_2^{j_2}\cdots y_n^{j_n},$$
$$\overline{\partial_h}(py_1^{i_1}y_2^{i_2}\cdots y_n^{i_n}x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}) = p' y_1^{i_1}y_2^{i_2}\cdots y_n^{i_n}x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n},$$

where p' denotes the usual derivative of p with respect to h.

We list some immediate properties:

1. $[\partial_r, \partial_s] = 0$ for $r, s \in \{x_1, \cdots, x_n, h, y_1, \cdots, y_n\}$.
2. $[\partial_{x_l}, x_l] = 1$ and $[\partial_{x_l}, r] = 0$ for
$$r \in \{x_1, \cdots, x_n, h, y_1, \cdots, y_n\} \setminus \{x_l\}.$$
3. $[\partial_{y_l}, y_l] = 1$ and $[\partial_{y_l}, r] = 0$ for
$$r \in \{x_1, \cdots, x_n, h, y_1, \cdots, y_n\} \setminus \{y_l\}.$$
4. $[\partial_h, h] = 1$, $[\partial_h, y_l] = -\partial_{x_l}$ and $[\partial_h, x_l] = 0$, for $1 \le l \le n$.
5. The above properties show that $\partial_{x_l}, \partial_{y_l} \in M_1$ and $\partial_h \in M_2$.
6. $\lambda_{x_l} - \rho_{x_l} = h\partial_{y_l}$, and $\rho_{y_l} - \lambda_{y_l} = h\partial_{x_l}$.
7. $\overline{\partial_h} = \partial_h + \sum_l \partial_{x_l}\partial_{y_l}$.
8. $[\overline{\partial_h}, x_l] = \partial_{y_l}$, $[\overline{\partial_h}, y_l] = 0$ and $[\overline{\partial_h}, h] = 1$.

Following the theorem 2.3.4 of [7], we show the following

Proposition 4.1.7. *Let characteristic of $\Bbbk$ be 0. The $\Bbbk$ -algebra $D_{\Bbbk}(H_n)$ is generated by left multiplications by elements of H_n and by*

$$\{\partial_{x_l}, \partial_{y_l}\}_{1 \le l \le n}, \partial_h.$$

Proof. Let $R_s \subset H_n$ denote the $\Bbbk$ -span of monomials in

$$x_1, \cdots, x_n, h, y_1, \cdots, y_n \text{ of } deg_2 \le s.$$

Claim : Let $D \in M_s$ be such that $D|_{R_s} = 0$. Then $D = 0$.
When $D \in M_0$ then $D = \rho_{D(1)}$ and hence the claim. Assume that we have proved the claim for $s \lneq i$ and fix $D \in M_i$ such that $D|_{R_j} = 0$ for some $j \ge i$. It is enough to show that for $c \in R_j, c' \in R_{j-1}$, we have $D(x_l c) = D(y_l c) = D(hc') = 0$. Note that $D(x_l c) = [D, x_l](c) + x_l D(c) = 0 + 0$. Similar argument for the variables y_l and the fact that $[M_i, h] \subset M_{i-2}$ complete the claim.

Let $A \subset D_{\Bbbk}(H_n)$ be the $\Bbbk$ -subalgebra generated by H_n and $\{\partial_{x_l}, \partial_{y_l}\}_{1 \le l \le n}, \partial_h$.
Claim : H_n is a simple A-module.
By assumption, the characteristic of the field is 0. Hence the claim

follows from the fact that given a $0 \neq c \in R_j$, there exists a $D \in \{\partial_{x_l}, \partial_{y_l}\}_{1 \leq l \leq n} \cup \{\partial_h\}$ such that $D(c) \neq 0$ and $D(c) \in R_{j-1}$.

Note that $Hom_A(H_n, H_n) = \Bbbk$. Now fix $0 \neq D \in M_s$. Then by the Jacobson Density Theorem, we can find $d \in A$ such that $d|_{R_s} = D|_{R_s}$. If $d \notin M_s$, then clearly, $d|R_s = 0$. Now, by the first claim, $d = D$. Hence the proposition. $\square$

Corollary 4.1.8. *If A_n denotes the nth-Weyl algebra (that is $h = 1$) over a field of characteristic 0, then $D_{\Bbbk}(A_n) = A_{2n}$*

Proof. By the above proposition, $D_{\Bbbk}(A_n)$ is generated by $\{\partial_{x_l}, \partial_{y_l}\}_{1 \leq l \leq n}$ and left multiplication by elements of A_n. Since $[x_i, y_j] = \delta_{i,j}$, we have $\partial_{x_l}, \partial_{y_l}$ are inner (A more general statement is true, due to Dixmier (Lemma 4.6.8 of [2]): All derivations on a Weyl algebra are inner). That is, $D_{\Bbbk}(A_n) = D^0_{\Bbbk}(A_n)$. By corollary 2.0.3, we have a surjection $A_n \bigotimes A_n^o \rightarrow D^0_{\Bbbk}(A_n)$. Note that A_n^o is isomorphic to A_n by mapping $x_l^o \mapsto -y_l$ and $y_l^o \mapsto x_l$. Also, $A_n \bigotimes A_n \cong A_{2n}$ by Corollary 1.2, page 122 of [1]. Thus, we have a surjection $A_{2n} \rightarrow D^0_{\Bbbk}(A_n)$. Now use the fact that A_{2n} is simple to complete the corollary. $\square$

Theorem 4.1.9. *Let characteristic of $\Bbbk$ be 0. The ring $D_{\Bbbk}(H_n)$ is generated by left multiplication by elements of H_n and by $\{\partial_h, \overline{\partial_h}\}$. That is, the ring $D_{\Bbbk}(H_n)$ is generated by two copies of $D_{\Bbbk}(\Bbbk[h])$ and inner derivations.*

Proof. By the properties following definitions 4.1.6 we see that ∂_h and $\overline{\partial_h}$ generate ∂_{x_l} and ∂_{y_l} for all l. The previous proposition completes the theorem. $\square$

Theorem 4.1.10. *Let $\Bbbk$ be a field of characteristic 0. The ring of differential operators on H_n is simple.*

Proof. Let $a \in D_{\Bbbk}(H_n)$. Then a can be written as a $\Bbbk$ -linear combination of monomials of the form $h^m \mathbf{x}^I \mathbf{y}^J \partial_h^s {\partial_{\mathbf{x}}}^K {\partial_{\mathbf{y}}}^L$ where

$$ {\partial_{\mathbf{x}}}^K = \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \cdots \partial_{x_n}^{k_n} $$

where $K = (k_1, k_2, \cdots, k_n)$ a multiindex. Let $\mathcal{I}$ be an ideal in $D_{\Bbbk}(H_n)$. Let $0 \neq a \in \mathcal{I}$. As $[\partial_h^s, h] = s\partial_h^{s-1}$ and the fact that h commutes with all the other generators, we can assume that ∂_h does not appear in the expression of a. Now use the fact that $[x_l^k \partial_{y_l}^s, y_l] = khx_l^{k-1}\partial_{y_l}^s + sx_l^k \partial_{y_l}^{s-1}$ and the fact that y_l commutes with all the other generators, to assume that in the expression of a, the generators x_l and ∂_{y_l} do not appear. Similarly, as $[y_l^k \partial_{x_l}^s, x_l] = -khy_l^{k-1}\partial_{x_l}^s + sy_l^k \partial_{x_l}^{s-1}$, we can assume that a

is a polynomial in h. Now use the fact that $[h^s, \partial_h] = sh^{s-1}$ to conclude that there is a non zero scalar in $\mathcal{I}$ and hence $\mathcal{I} = D_{\Bbbk}(H_n)$. □

4.2. **Characteristic of $\Bbbk = p \neq 0$.** In this case, the centre is the polynomial ring in $2n+1$ variables

$$R := \Bbbk[h, x_1^p, x_2^p, \cdots, x_n^p, y_1^p, y_2^p, \cdots, y_n^p].$$

Theorem 2 of [8] shows that the nth-Weyl algebra is Azumaya over its centre when characteristic of $\Bbbk$ is nonzero. The same proof works to show that H_n is Azumaya over its centre. Now we refer to the section on Azumaya algebra to claim:

Theorem 4.2.1. $D_{\Bbbk}(H_n) = (H_n \bigotimes_R H_n^o) \cdot D_{\Bbbk}(R) \cdot (H_n \bigotimes_R H_n^o)$.

In [9], the differential operators on polynomial ring in one variable, on a field of nonzero characteristic has been studied. In particular it is shown that $D_{\Bbbk}(R)$ is simple.

Corollary 4.2.2. *Let $\Bbbk$ be a field of non zero characteristic. The ring $D_{\Bbbk}(H_n)$ is simple.*

5. Concluding remarks and acknowledgements

This work suggests that if R is the centre of A, and if there is a way of embedding $D_{\Bbbk}(R)$ into $D_{\Bbbk}(A)$, then $D_{\Bbbk}(R)$ generates $D_{\Bbbk}(A)$ as an A^e-module. Further natural questions are to find differential operators on the enveloping algebras of Lie algebras.

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